The intersection cohomology Hodge module on toric varieties (joint w/ Hyunsuk Kim)

§1. Summary

(1) X variety$/\mathbb{C}$. Have intersection cohomology complex IC_X, perverse sheaf.
Good substitute for $\mathbb{Q}_X$ when X singular.

(2) IC_X has a refined structure as <u>Hodge module</u> IC_X^H.

(3) Have graded de Rham complexes $gr_p DR(IC_X^H)$ $\in D^b_{coh}(X)$

(4) Can we calculate "explicitly" for toric varieties?

(5) (Kim-V '24) Yes, numerically.
Generating function of $gr_p DR(IC_X^H)$
= (Generating function of IC_X)(Simple correction terms).

§2. Toric varieties

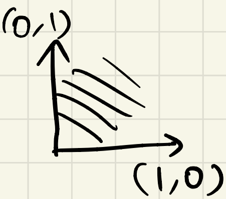
N = Free ab. gp of rk n $(\simeq \mathbb{Z}^n)$ Lattice

$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ Dual lattice.

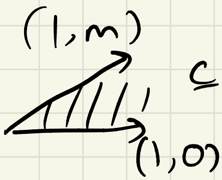
(Nice) cone $\sigma \subseteq N \otimes \mathbb{R} \rightsquigarrow$ n-dim. affine 'toric' variety X_σ

$[\, X_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M] \,]$

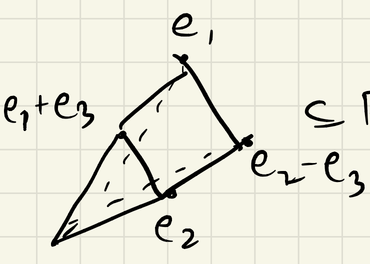
Ex:



$\subseteq \mathbb{R}^2 \longleftrightarrow \mathbb{A}^2$ $\quad (\lambda_1, \lambda_2).(x, y) = (\lambda_1 x, \lambda_2 y)$



$\subseteq \mathbb{R}^2 \longleftrightarrow \{x^m = yz\}$



$\subseteq \mathbb{R}^3 \longleftrightarrow \{xy = zw\}$

Slogan: Geometry of $X_\sigma \longleftrightarrow$ Convex geometry/ Combinatorics of σ

<u>Fact</u> : X_σ smooth $\Leftrightarrow$ σ generated by part of a basis of N

X_σ finite quot. sing $\Leftrightarrow$ σ " " " " of $N \otimes \mathbb{R}$

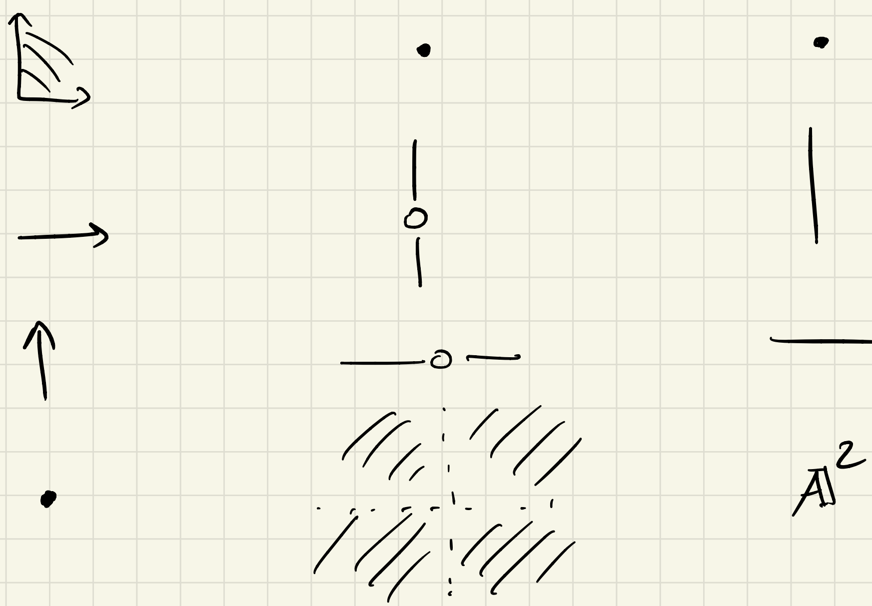
"simplicial"

$$IC_{X_\sigma} \simeq \mathbb{Q}_{X_\sigma}[n].$$

<u>Fact</u> : (1) Torus $T \simeq (\mathbb{C}^*)^n \underset{\text{dense open}}{\subseteq} X_\sigma$ & $T \curvearrowright X_\sigma$

(2) $\{\text{Faces of } \sigma\} \xleftrightarrow[\text{rever}]{\text{incl.}} \{T\text{-orbits}\} \longleftrightarrow \{T\text{-invar. subvar}\}$

$$\tau \longleftrightarrow O_\tau \longleftrightarrow \overline{O_\tau} = S_\tau$$



§3. Hodge modules (M. Saito)

A Hodge module is $(\mathcal{M}, F, K)$ where

(1) $\mathcal{M}$ = $\mathcal{D}$-module

(2) F = Hodge filtration on $\mathcal{M}$

(3) K = Perverse sheaf on X

Ex: • X smooth, $(\mathcal{O}_X, F, \mathbb{Q}_X[n])$

$$F_k \mathcal{O}_X = \begin{cases} 0 & \text{if } k < 0 \\ \mathcal{O}_X & \text{if } k \geq 0 \end{cases}$$

• X singular (IC_X^H, F, IC_X)

$$DR_X(\mathcal{M}) = [\underset{-n}{\mathcal{M}} \xrightarrow{\nabla} \underset{-(n-1)}{\Omega^1_X \otimes \mathcal{M}} \xrightarrow{\nabla} \Omega^2_X \otimes \mathcal{M} \to \cdots \underset{0}{\Omega^n_X \otimes \mathcal{M}}]$$

$$F_k DR(\mathcal{M}) = [F_k\mathcal{M} \to \Omega^1_X \otimes F_{k+1}\mathcal{M} \to \cdots \to \Omega^n_X \otimes F_{k+n}\mathcal{M}]$$

$$gr^F_k DR(\mathcal{M}) = [gr^F_k \mathcal{M} \to \Omega^1_X \otimes gr^F_{k+1}\mathcal{M} \to \cdots \to \Omega^n_X \otimes gr^F_{k+n}\mathcal{M}]$$

$$\in D^b_{coh}(X)$$

Ex: • X smooth

$$DR(\mathcal{O}_X) = [\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \to \Omega^n_X]$$

$$F_{-k} DR(\mathcal{O}_X) = [0 \to 0 \cdots \to 0 \to \Omega^k_X \to \cdots \Omega^n_X]$$

$$gr^F_{-k} DR(\mathcal{O}_X) = \Omega^k_X[n-k]$$

• X simplicial toric IC^H_X

$$DR(IC^H_X) = [\mathcal{O}_X \to \Omega^{[1]}_X \to \cdots \Omega^{[n]}_X]$$

reflexive differentials

$$gr^F_{-k} DR(\mathcal{O}_X) = \Omega^{[k]}_X[n-k].$$

Rmk: • Kebekus - Schnell '21 use this to prove results about extension of holo forms. (Related: Park '24, Tighe '24)

• Popa - Shen - Vo '24, Popa - Park '24 call these 'intersection Du Bois complexes' $\underline{I\Omega}^p_X$.

$$\underline{\Omega}^p_X \longrightarrow \underline{I\Omega}^p_X$$

p^{th} Du Bois cplx

§4. Main result

IC_X & $gr_k DR(IC^H_X) \in D^b_{con}(X)$

$\mathcal{H}^j(IC_X)$ constant on orbits O_τ. $\tau \underset{face}{\subseteq} \sigma$.

$\widetilde{H}_{X,\tau} := \underbrace{q^{n-d_\tau}}\sum_j \dim_{\mathbb{C}} H^j(IC_X)_{x_\tau} q^j \qquad x_\tau \in O_\tau$

$\{\widetilde{H}_{X,\tau}\}_{\tau \underset{face}{\subset} \sigma}$ encodes the info of IC_X

$\mathcal{H}^l(gr_k DR(IC^H_X))$ = Torus-equiv. sheaf

[<u>Aside</u>: Dual lattice $M \simeq Hom(T, \mathbb{C}^*)$ = Characters

$X = Spec\, R$ affine toric, F = T-equiv. R-module

$F = \bigoplus_{u \in M} F_u \leftarrow$ u-eigenspace

For $u \in$ Dual face of τ, F_u's isomorphic

$dR_{X,\tau}(K,L) := \sum \dim \mathcal{H}^l(gr_k DR(IC^H_X))_u K^k L^l$ $\quad u \in$ Dual face of τ

$\{dR_{X,\tau}\}_{\tau \underset{face}{\subseteq} \sigma}$ encodes the info of $\{gr_k DR(IC^H_X)\}_{k \in \mathbb{Z}}$

Thm: (Kim - V '24) $\forall \tau \underset{\text{face}}{\subseteq} \sigma$

$$dR_{X,\tau}(K,L) = \widetilde{H}_{X,\tau}(K^{-\frac{1}{2}}L)\, K^{-\frac{d_\tau}{2}} (K^{-1}+L^{-1})^{n-d_\tau}$$

$$d_\tau = \dim(\tau)$$

Moreover $\widetilde{H}_{X,\tau}$ and $dR_{X,\tau}$ can be computed explicitly in an algorithmic.

§5. Proof sketch

Thm: (Toric DT: dCMM '18) $\pi : Y \to X = X_\sigma$ proper birational toric map, Y = simplicial.

$$R\pi_* \underbrace{IC_Y}_{\substack{\wr\wr \\ \mathbb{Q}_Y[n]}} \simeq \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\mu \subseteq \sigma} IC_{S_\mu}^{\oplus s_{\mu,j}} [-j]$$

$\mathcal{H}^j(\quad)_{x_\tau}$ both sides $\qquad x_\tau \in O_\tau$

(Saito) $$\pi_* IC_Y^H \simeq \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\mu \subseteq \sigma} IC^H_{S_\mu}(-)^{\oplus s_{\mu,j}} [-j]$$

$$\begin{aligned} gr_{-p} DR(\pi_* IC_Y^H) &= R\pi_* \, gr_{-p} DR(IC_Y^H) \\ &= R\pi_* \, \Omega_Y^{[p]}[n-p] \end{aligned}$$

$$\widetilde{F}_\tau(q) := q^{-d_\tau} \sum_j \dim H^j(\pi^{-1}(x_\tau), \mathbb{Q}) \cdot q^j$$

$\widetilde{H}_{x,\tau}$

$$D_\tau(q) = \sum s_{\tau,j}\, q^j$$

$$\Omega_\tau(K,L) = \sum_{k,l} \dim_{\mathbb{C}} \left(R^l \pi_* \Omega^k_Y\right)_u K^k L^l \quad u \in \text{Dual face of } \tau$$

$dR_{x,\tau}$

Pf: (1) $\widetilde{F}_\tau \longleftrightarrow \widetilde{H}_{x,\tau}$ and D_τ

$\Omega_\tau \longleftrightarrow dR_{x,\tau}$ and D_τ

(2) Choose 'barycentric res' of σ,

get $\widetilde{F}_\tau \Longleftrightarrow \Omega_\tau$

$$\rightsquigarrow dR_{x,\tau}(K,L)' = \widetilde{H}_{x,\tau}(K^{-\frac{1}{2}}L)\, K^{\frac{-d_\tau}{2}} (K^{-1}+L^{-1})^{n-d_\tau}$$

$\boxed{/\!/}$